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SIMPLE PENDULUM: PERIOD DEPENDENT ON AMPLITUDE OF OSCILLATION

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ABSTRACT: With a pedagogical aim suited for the undergraduate, in this study, the differential equations of motion that characterize and determine the motion of a simple pendulum were obtained considering small and large amplitudes of oscillation. These differential equations were solved through differential equation solution methods, numerical, expansion of functions and integrations. The solutions obtained using the different methods were compared. It was possible to verify, both experimentally and theoretically, that for the oscillatory movement of the simple pendulum, its oscillation period increases and its angular frequency decreases with the increase of the oscillation amplitude. The validity range of the approximation for small ranges of motion was also determined. It was verified that the theoretical and experimental results present a good agreement for angles smaller than 55°. The experimental measurements were made with "a low-cost home-built" equipment. It should be noted that some factors can generate discrepancies between experimental and theoretical results.

KEYWORDS: Oscillations. Simple Pendulum. Large Amplitudes. Low-Cost Home-Built Equipment.

PÊNDULO SIMPLES: PERÍODO DEPENDENTE DA AMPLITUDE DE OSCILAÇÃO

RESUMO: Com objetivo pedagógico voltado para o nível de graduação, neste estudo foram obtidas as equações diferenciais do movimento que caracterizam e determinam o movimento de um pêndulo simples considerando pequenas e grandes amplitudes de oscilação. Estas equações diferenciais foram resolvidas utilizando métodos de solução de equações diferenciais, métodos numéricos, expansão de funções e integrações. As soluções obtidas usando os diferentes métodos foram comparadas. Foi possível verificar, tanto experimental quanto teoricamente, que para o movimento oscilatório do pêndulo simples, seu período de oscilação aumenta e sua frequência angular diminui com o aumento da amplitude de oscilação. A faixa de validade da aproximação para pequenas amplitudes de movimento também foi determinada. Verificou-se que os resultados teóricos e experimentais apresentam uma boa concordância para ângulos de oscilação menores que 55°. As medições experimentais foram feitas com um equipamento de baixo custo construído pelos autores. Cabe ressaltar que alguns fatores podem gerar discrepâncias entre resultados experimentais e teóricos.

PALAVRAS-CHAVE: Oscilações. Pêndulo Simples. Grandes Amplitudes. Experimento de Baixo Custo.

INTRODUCTION

Periodic and oscillatory motions are of great interest due to their diverse applications in the field of physics and engineering. A detailed study of the physics involved in these periodic movements is extremely important for a description of the phenomenon and for determining the laws that govern its movement (LANDAU, 1976). The knowledge of these laws brings a better clarity about the phenomenon in question, giving support to possible applications. An important example for an introductory study of oscillations is the 'simple pendulum' (NELSON, 1986).

The word 'pendulum' comes from the Latin "pendulus", which means "hanging". A pendulum is a weight suspended from a pivot so that it can swing freely, Figure 1. When the pendulum is removed out of its equilibrium position and then released (with null initial velocity), it starts to oscillate. It is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position (Symon, 1971). The restoring force acting on the pendulum's mass causes it to oscillate about the equilibrium position. The period T of a vibratory motion is the time required for a complete to-and-fro motion or oscillation. In a complete oscillation the vibrating body moves from the equilibrium position, to the other end of the path, and back to the equilibrium position ready to repeat the cycle. The frequency f of the vibratory motion is the number of complete oscillations per unit time. The frequency is the reciprocal of the period: $f = 1/T$. The amplitude of a vibratory motion is the maximum displacement from the equilibrium position (RODRIGUES, 2020).

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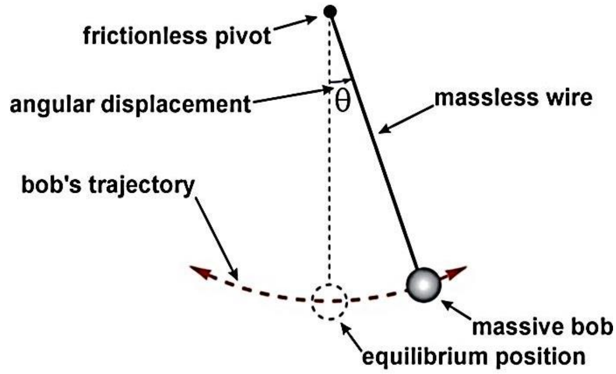
Figure 1 – Simple gravity pendulum.



Source: authors.

From the first scientific investigations of the pendulum around 1602 by Galileo Galilei (Newton, 2004), the regular motion of pendulums was used for timekeeping, and was the world's most accurate timekeeping technology until the 1930s. The pendulum clock invented by Christiaan Huygens (ANDRIESSE, 2005) in 1658 became the world's standard timekeeper, used in offices and homes for 270 years, and achieved accuracy of about one second per year before it was superseded as a time standard by the quartz clock in the 1930s (MARRISON, 1948). Pendulums are also used in scientific instruments such as accelerometers, seismometers, and gravimeters to measure the acceleration of gravity in geo-physical surveys.

In this paper we obtain the period of the simple pendulum for small oscillations (Section 2) and for large oscillations (Section 3). Section 4 presents experimental measurements performed using a "low-cost home-built" equipment. Also in Section 4, comparisons are made between experimental and theoretical results. Section 5 is reserved for some final comments.

SIMPLE PENDULUM: APPROXIMATION FOR SMALL OSCILLATIONS

We consider a simple pendulum with frictionless pivot, no air resistance, and inextensible and massless wire. Figure 2 shows a scheme of the forces acting on mass m, and their decompositions.

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Two forces act on the particle of mass m: the weight $\vec{P}$ and the tension $\vec{T}$. Weight $\vec{P}$ can be written as follows (THORNTON, 2020)

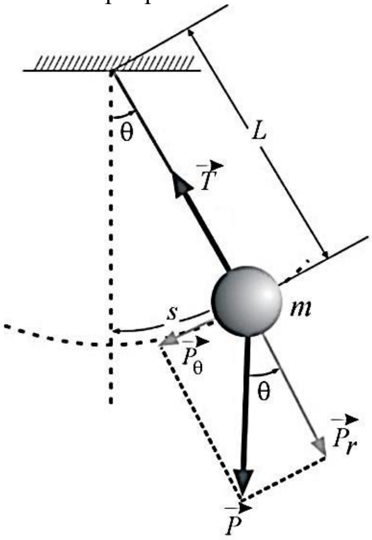
$$\vec{P} = \vec{P}_\theta + \vec{P}_r = -P_\theta \hat{\theta} + P_r \hat{r}, \qquad (1)$$

where

$$\sin\theta = \frac{P_\theta}{P} \Rightarrow P_\theta = P\sin\theta = mg\sin\theta, \qquad (2)$$

$$\cos\theta = \frac{P_r}{P} \Rightarrow P_r = P\cos\theta = mg\cos\theta. \qquad (3)$$

Figure 2 – Decomposition of forces in a simple pendulum.



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Using Newton's 2nd Law (GOLDSTEIN, 1980)

$$\vec{F} = \vec{P} + \vec{T} = -P_\theta \hat{\theta} + P_r \hat{r} - T\hat{r}, \qquad (4)$$

$$F_r \hat{r} + F_\theta \hat{\theta} = -P_\theta \hat{\theta} + P_r \hat{r} - T\hat{r}. \qquad (5)$$

Thus

$$F_r \hat{r} = (P_r - T)\hat{r}, \qquad (6)$$

$$F_\theta \hat{\theta} = -P_\theta \hat{\theta}. \qquad (7)$$

Eq. (4) can be written as follows

$$m\vec{a} = -P_\theta \hat{\theta} + P_r \hat{r} - T\hat{r}. \qquad (8)$$

Using acceleration in polar coordinates

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}. \qquad (9)$$

For the simple pendulum we have

$$r = L \Rightarrow \dot{r} = 0 \Rightarrow \ddot{r} = 0,$$

and

$$\vec{a} = -r\dot{\theta}^2\hat{r} + r\ddot{\theta}\hat{\theta}. \qquad (10)$$

Inserting Eq. (10) into Eq. (8)

$$m(-L\dot{\theta}^2\hat{r} + L\ddot{\theta}\hat{\theta}) = -P_\theta\hat{\theta} + P_r\hat{r} - T\hat{r}.$$

Separating this last equation into two (since $\hat{r}$ and $\hat{\theta}$ are orthogonal vectors)

$$-mL\dot{\theta}^2\hat{r} = (P_r - T)\hat{r}, \qquad (11)$$

$$mL\ddot{\theta}\hat{\theta} = -P_\theta\hat{\theta}. \qquad (12)$$

Inserting Eq. (2) into Eq. (12) yields

$$mL\ddot{\theta} = -mg\sin\theta,$$

$$\ddot{\theta} = -\frac{g}{L}\sin\theta,$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0. \qquad (13)$$

For small angles we can approximate: $\sin\theta \approx \theta$. Then, equation (13) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \qquad (14)$$

Defining

$${\omega_0}^2 = \frac{g}{L}, \qquad (15)$$

we have

$$\frac{d^2\theta}{dt^2} + {\omega_0}^2\theta = 0. \qquad (16)$$

Equation (16) is a second order differential equation, homogeneous and with constant coefficients, that is, it is an equation of the type (RODRIGUES, 2017)

$$a\frac{d^2Y(t)}{dt^2}+b\frac{dY(t)}{dt}+cY(t)=0. \qquad (17)$$

The solution to equation (17) is

$$Y(t)=c_1e^{x_1t}+c_2e^{x_2t}, \text{if } x_1\neq x_2, \qquad (18)$$

or

$$Y(t)=c_1e^{xt}+c_2te^{xt}, \text{if } x_1=x_2=x, \qquad (19)$$

where x is the roots of the characteristic polynomial

$$ax^2+bx+c=0.$$

Comparing Eq. (17) with Eq. (16) we have

$$a=1; b=0 \text{ and } c={\omega_0}^2,$$

thus

$$x^2+0+{\omega_0}^2=0\Rightarrow x^2=-{\omega_0}^2\Rightarrow x=\pm\sqrt{-{\omega_0}^2}.$$

By Eq. (15) we note that ${\omega_0}^2>0$, since g and L are positive quantities. Thus

$$x=\pm i\omega\Rightarrow\begin{cases}x_1=+i\omega_0\\x_2=-i\omega_0\end{cases}$$

Substituting these values of x_1 and x_2 in Eq. (18), we have

$$\theta(t)=c_1e^{i\omega_0t}+c_2e^{-i\omega_0t}.$$

Using Euler's formula (Rodrigues, 2017), the previous equation becomes

$$\theta(t)=c_1[\cos(\omega_0t)+i\sin(\omega_0t)]+c_2[\cos(\omega_0t)-i\sin(\omega_0t)],$$

$$\theta(t)=(c_1+c_2)\cos(\omega_0t)-(c_2-c_1)i\sin(\omega_0t). \qquad (20)$$

Defining

$$c_1+c_2=A\cos\beta, \qquad (21)$$

and

$$(c_1-c_2)i=A\sin\beta, \qquad (22)$$

where β is a constant, Eq. (20) takes the form

$$\theta(t)=A[\cos\beta.\cos(\omega_0t)-\text{sen}\beta.\sin(\omega_0t)]. \qquad (23)$$

Using the trigonometric relationship

$$\cos(\beta+\delta)=\cos(\beta).\cos(\delta)-\sin(\beta).\sin(\delta),$$

with $\delta=\omega_0t$, Eq. (23) takes the form

$$\theta(t)=A\cos(\omega_0t+\beta). \qquad (24)$$

In Eq. (24) it is possible to identify that: A is the amplitude of motion, ω_0 is the angular frequency and β is the initial phase. Eq. (24) is the solution for $\theta(t)$ for the physical pendulum problem, considering small oscillations. Note that adopting a different definition in Eqs. (21) and (22), for example, $c_1 + c_2 = A\sin\beta$ and $(c_2 - c_1)i = A\cos\beta$, the following solution would have been obtained

$$\theta(t) = A\sin(\omega_0 t + \beta), \qquad (25)$$

which is physically equivalent to Eq. (24).

By Eq. (15) the angular frequency for the simple pendulum, considering small oscillations ($\text{sen}\theta \approx \theta$) is

$$\omega_0 = \sqrt{\frac{g}{L}}, \qquad (26)$$

and remembering that $\omega_0 = 2\pi/T_0$, period T_0 is

$$T_0 = 2\pi\sqrt{\frac{L}{g}}. \qquad (27)$$

Note from Eqs. (26) and (27) that for small oscillations the angular frequency ω_0 and the period T_0 are independent of the amplitude of oscillation of the movement.

SIMPLE PENDULUM WITH LARGE AMPLITUDES

The problem of simple pendulum motion for large amplitudes can be treated in terms of the total mechanical energy of the system (GOLDSTEIN, 1980)

$$E = K + V, \qquad (28)$$

where K is the kinetic energy and V is the potential energy of the system. Note from Figure 3 that

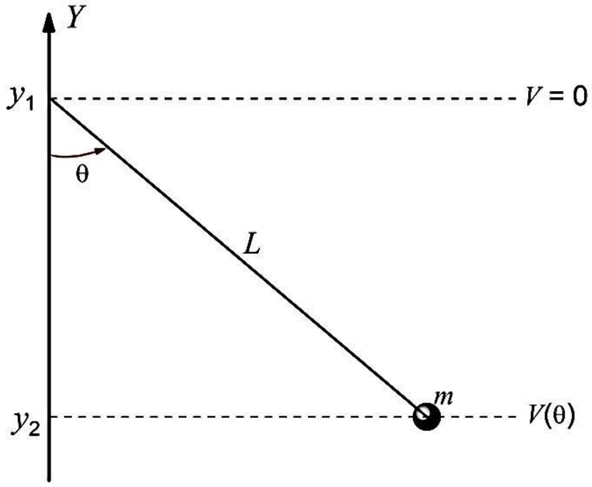
$$V = mg\Delta y = mg(y_2 - y_1),$$

and assuming y_1 is at the origin of the Y axis

$$V = mg(-L\cos\theta - 0),$$

$$V(\theta) = -mgL\cos\theta. \qquad (29)$$

Figure 3 – Potential energy $V(\theta)$.



Source: authors.

The kinetic energy K is given by

$$K = \frac{m}{2}v^2.$$

Velocity v can be written in terms of the variable θ as: $v = L\dot{\theta}$. Thus, the previous expression for the kinetic energy K of the system can be expressed by

$$K = \frac{m}{2}L^2\dot{\theta}^2. \qquad (30)$$

Inserting Eqs. (29) and (30) in Eq. (28) we have

$$E = \frac{m}{2}L^2\dot{\theta}^2 - mgL\cos\theta. \qquad (31)$$

We will take the simplification that there are no dissipative forces in the system and therefore the total mechanical energy E is constant. Figure 4 shows the curve of potential energy *versus* θ. It is verified that for $-mgL < E < mgL$ the movement is oscillatory. If mass m were attached to a rigid rod (with negligible mass) of length L instead of a wire, the pendulum could rotate in a circle if $E > mgL$. Even so, this movement would still be periodic, as the pendulum would make a complete revolution each time θ increases by 2π.

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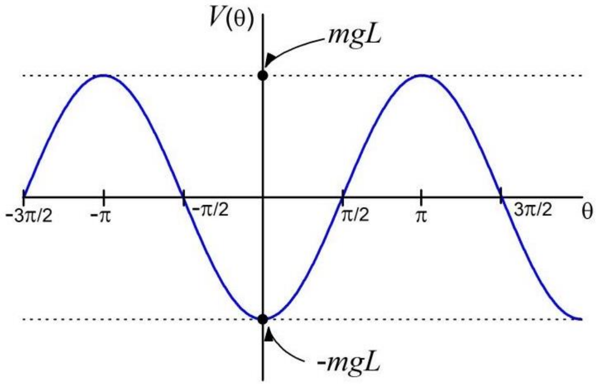
Figure 4 – Potential energy for the simple pendulum as a function of angle θ.



Source: authors.

In this study we will consider only the case of a simple pendulum composed of a wire of length L and a loose mass m, which is released from an initial angle smaller than $\pi/2$. Eq. (31) can be written as follows

$$\frac{m}{2}L^2\dot{\theta}^2 = E + mgL\cos\theta,$$

$$\dot{\theta}^2 = \frac{2E}{mL^2} + \frac{2mgL}{mL^2}\cos\theta = \frac{2g}{L}\left(\frac{E}{mgL} + \cos\theta\right),$$

$$\dot{\theta} = \sqrt{\frac{2g}{L}}\sqrt{\frac{E}{mgL} + \cos\theta},$$

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}}\sqrt{\frac{E}{mgL} + \cos\theta},$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\frac{E}{mgL} + \cos\theta}} = \int_0^t \sqrt{\frac{2g}{L}}\,dt,$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\frac{E}{mgL} + \cos\theta}} = \sqrt{\frac{2g}{L}}\,t. \qquad (32)$$

The period of movement can be obtained by solving the integral between appropriate limits. When the motion is oscillatory, that is, $E < mgL$, the maximum value of θ, which we will call α, is given according to Eq. (31) by

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$$-mgL\cos\alpha = E. \qquad (33)$$

Thus, Eq. (32) becomes

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = \sqrt{\frac{2g}{L}}\, t. \qquad (34)$$

The angle θ oscillates between the limits $-\alpha < \theta < \alpha$. Let us consider the trigonometric relationship

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B),$$

and with $A = B = \gamma/2$

$$\cos(\gamma/2 + \gamma/2) = \cos(\gamma/2)\cos(\gamma/2) - \sin(\gamma/2)\sin(\gamma/2),$$

$$\cos(\gamma) = \cos^2(\gamma/2) - \sin^2(\gamma/2),$$

$$\cos(\gamma) = 1 - \sin^2(\gamma/2) - \sin^2(\gamma/2),$$

$$\cos(\gamma) = 1 - 2\sin^2(\gamma/2).$$

Therefore

$$\cos(\theta) = 1 - 2\sin^2(\theta/2),$$

$$\cos(\alpha) = 1 - 2\sin^2(\alpha/2),$$

and

$$\cos\theta - \cos\alpha = 1 - 2\sin^2(\theta/2) - 1 + 2\sin^2(\alpha/2),$$

$$\cos\theta - \cos\alpha = 2[\sin^2(\alpha/2) - \sin^2(\theta/2)]. \qquad (35)$$

Inserting Eq. (35) into Eq. (34) we have

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2}\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}} = \sqrt{\frac{2g}{L}}\, t,$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}} = 2\sqrt{\frac{g}{L}}\, t,$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\sin^2\left(\frac{\alpha}{2}\right)\left[1 - \left(\frac{\sin(\theta/2)}{\sin(\alpha/2)}\right)^2\right]}} = 2\sqrt{\frac{g}{L}}\, t,$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{\sin\left(\frac{\alpha}{2}\right)\sqrt{\left[1 - \left(\frac{\sin(\theta/2)}{\sin(\alpha/2)}\right)^2\right]}} = 2\sqrt{\frac{g}{L}}\, t. \qquad (36)$$

Remembering that the angle θ oscillates between the limits $\pm\alpha$, and introducing a new variable φ that varies from 0 to 2π in a complete cycle of oscillation of θ as follows

$$\sin\varphi = \frac{\sin(\theta/2)}{a}, \qquad (37)$$

where

$$a = \sin(\alpha/2). \qquad (38)$$

Note that

$$\frac{d}{d\theta}\sin\varphi = \frac{\cos(\theta/2)}{2a},$$

$$\cos\varphi\frac{d\varphi}{d\theta} = \frac{1}{2a}\cos(\theta/2),$$

$$\frac{2a\cos\varphi}{\cos(\theta/2)}d\varphi = d\theta. \qquad (39)$$

By Eq. (37)

$$\sin^2\varphi = \frac{\sin^2(\theta/2)}{a^2},$$

$$a^2\sin^2\varphi = 1 - \cos^2(\theta/2),$$

$$\cos^2(\theta/2) = 1 - a^2\sin^2\varphi,$$

$$\cos(\theta/2) = \sqrt{1 - a^2\sin^2\varphi}. \qquad (40)$$

Thus, Eq. (39) takes the form

$$\frac{2a\cos\varphi}{\sqrt{1 - a^2\sin^2\varphi}}d\varphi = d\theta. \qquad (41)$$

Using Eqs. (37), (38) and (41), Eq. (36) becomes

$$\int_0^\varphi \frac{1}{a\sqrt{1-\sin^2\varphi}}\frac{2a\cos\varphi}{\sqrt{1-a^2\sin^2\varphi}}d\varphi = 2\sqrt{\frac{g}{L}}t,$$

$$\int_0^\varphi \frac{1}{\sqrt{\cos^2\varphi}}\frac{\cos\varphi}{\sqrt{1-a^2\sin^2\varphi}}d\varphi = \sqrt{\frac{g}{L}}t,$$

$$\int_0^\varphi \frac{d\varphi}{\sqrt{1-a^2\mathrm{sen}^2\varphi}} = \sqrt{\frac{g}{L}}t. \qquad (42)$$

This integral has the standard form of elliptic integrals (GRADSTHTEYN, 2007)

$$\int \frac{dx}{\Delta} = F(x, a),$$

where $\Delta = \sqrt{1 - a^2 \sin^2 x}$ and $F(x, a)$ is called the elliptic integral of the first kind (WEISSTEIN, 2021). The integral of Eq. (42) can be solved by expanding the denominator into a power series, and then integrating term by term. Note that

$$(1 - x)^{-1/2} = 1 + \frac{x}{2} - \cdots \ (-1 < x \leq 1).$$

then

$$(1 - a^2 \sin^2\varphi)^{-1/2} = 1 + \frac{a^2 \sin^2\varphi}{2} + \cdots$$

and the integral in Eq. (42) becomes

$$\int_0^{\varphi} \left(1 + \frac{a^2 \sin^2\varphi}{2} + \cdots\right) d\varphi = \sqrt{\frac{g}{L}}\, t,$$

$$\varphi + \frac{a^2}{8}(2\varphi - \sin 2\varphi) + \cdots = \sqrt{\frac{g}{L}}\, t. \qquad (43)$$

The period of movement is obtained by setting $\varphi = 2\pi$ in Eq. (43)

$$2\pi + \frac{a^2}{8}[4\pi - \sin(4\pi)] + \cdots = \sqrt{\frac{g}{L}}\, T,$$

$$2\pi + 2\pi \frac{a^2}{4} + \cdots = \sqrt{\frac{g}{L}}\, T,$$

$$T = 2\pi \sqrt{\frac{L}{g}} \left(1 + \frac{a^2}{4} + \cdots\right). \qquad (44)$$

Note that in Eq. (44) the term $2\pi\sqrt{L/g}$ is the period T_0 of the simple pendulum for small oscillations (review Eq. (27) in Section 2). Thus, Eq. (44) becomes

$$T \approx T_0 \left(1 + \frac{a^2}{4}\right). \qquad (45)$$

Remembering that by Eq. (38), $a = \sin(\alpha/2)$, where α is the limit angle of oscillation of θ. Note by Eq. (45) that when the oscillation amplitude becomes large, the period becomes slightly longer than for small oscillations. The angular frequency ω of the motion can be obtained from Eq. (45) as follows

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{g}{L}}\left(1 + \frac{a^2}{4} + \cdots\right)^{-1},$$

and using the expansion

$$(1+x)^{-1} = 1 - x + x^2 - \cdots \quad (-1 < x < 1)$$

the angular frequency ω takes the form

$$\omega \approx \sqrt{\frac{g}{L}}\left(1 - \frac{a^2}{4}\right). \qquad (46)$$

Note that in Eq. (46) the term $\sqrt{g/L}$ is the angular frequency ω_0 of the simple pendulum for small oscillations (review Eq. (26) in Section 2). Thus, Eq. (46) becomes

$$\omega \approx \omega_0\left(1 - \frac{a^2}{4}\right), \qquad (47)$$

evidencing that for large amplitudes the angular frequency is smaller than in the situation of small oscillation amplitudes.

The period of oscillation T of the simple pendulum can also be obtained without performing a power series expansion in the denominator of Eq. (42). By Eq. (42)

$$\sqrt{\frac{g}{L}}T = \frac{2\pi}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{1 - a^2\sin^2\varphi}},$$

$$T = 2\pi\sqrt{\frac{g}{L}}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{1 - a^2\sin^2\varphi}},$$

$$T = \frac{T_0}{2\pi}\int_0^{2\pi}\frac{d\varphi}{\sqrt{1 - a^2\sin^2\varphi}}. \qquad (48)$$

The integral of Eq. (48) can be solved numerically. For this, some mathematical software can be used, such as Maple (GEDDES, 2021) or Mathematica (WOLFRAM, 1991). Table 1 shows the results obtained for the ratio T/T_0 using equations (45) and (48). Numerical values were obtained using Mathematica WOLFRAM 11.0 (WOLFRAM, 1991).

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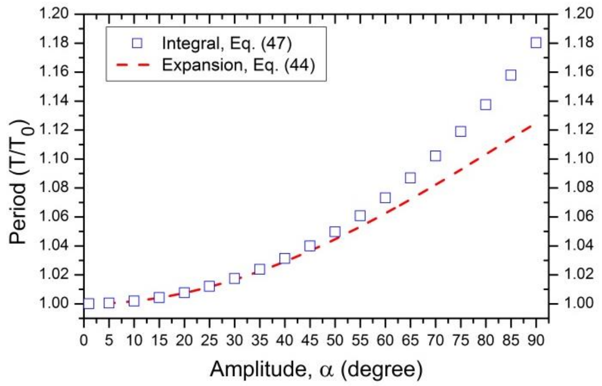
Table 1 – Results obtained for period of oscillation using Eqs. (45) and (48).

Oscillation amplitude (angle α)	T/T_0 (Eq. 45)	T/T_0 (Eq. 48)
1°	1.000019	1.000019
5°	1.000475	1.000476
10°	1.001899	1.001907
15°	1.004259	1.004300
20°	1.007538	1.007669
25°	1.011711	1.012030
30°	1.016746	1.017408
35°	1.022605	1.023833
40°	1.029244	1.031340
45°	1.036611	1.039973
50°	1.044651	1.049782
55°	1.053302	1.060829
60°	1.062500	1.073182
65°	1.072172	1.086922
70°	1.082247	1.102144
75°	1.092647	1.118959
80°	1.103293	1.137492
85°	1.114105	1.157894
90°	1.125000	1.180340

Source: authors.

Figure 5 shows the results presented in Table 1 for the period T/T_0. The square symbols were obtained using Eq. (48) and the dashed line using Eq. (45). Note that period T is greater than period T_0 and that as the amplitude of oscillation (angle α) increases, the discrepancy between period T and T_0 also increases. For an oscillation amplitude of 90° this difference is approximately 18% using Eq. (48) and 12% using Eq. (45). It is verified that the value of the period obtained using Eq. (48) is greater than that obtained using Eq. (45), and that the discrepancy between the two values increases with increasing amplitude of oscillation.

Figure 5 – Dependence on the amplitude of the ratio T/T_0. Square symbol: Eq. (48). Dashed line: Eq. (45).



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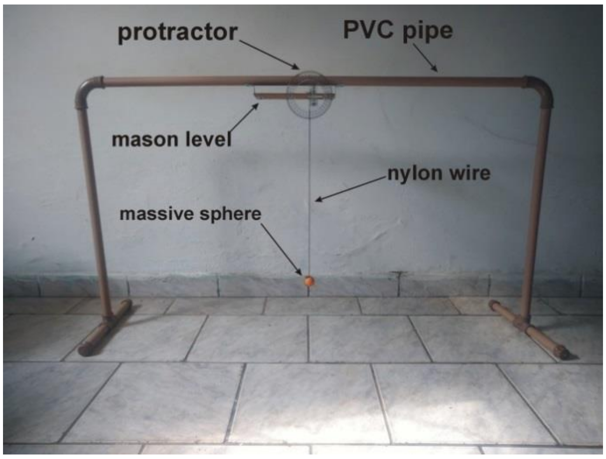
EXPERIMENTAL MEASUREMENTS

The experimental measurements were made with “a low-cost home-built” equipment: PVC pipes, an attached protractor to measure the angle at which the pendulum was released, a solid sphere, nylon wire, and a mason level were used, Figure 6. The length of the pendulum is 50 cm. The timing was performed with a cell phone stopwatch. The time Δt spent by the pendulum to perform ten complete oscillations ($N = 10$) for each angle was measured. Period T is: $T = \Delta t / N$. Table 2 shows the experimental results obtained. The first column of Table 2 is the starting angle at which the pendulum was released. The second column is the time taken for the pendulum to perform 10 complete oscillations. The third column is the period obtained for these 10 oscillations. Time measurements are in seconds and angle measurements are in degrees. From 5° the angle was varied from 5° in 5° up to the limit of 90°. It can be seen from Table 2 that when the oscillation amplitude is increased, the period T also increases.

Figure 6 – Equipment built to measure the period of the simple pendulum.



Fonte: Os autores.

Table 2 – Experimental measurements for the simple pendulum.

Oscillation amplitude (angle α)	Δt (s)	$T = \Delta t / N$ (s)
1°	14.27	1.427
5°	14.27	1.427
10°	14.27	1.427
15°	14.30	1.430
20°	14.30	1.430
25°	14.34	1.434
30°	14.57	1.457

35°	14.61	1.461
40°	14.65	1.465
45°	14.77	1.477
50°	14.83	1.483
55°	15.05	1.505
60°	15.12	1.512
65°	15.24	1.524
70°	15.26	1.526
75°	15.30	1.530
80°	15.51	1.551
85°	15.58	1.558
90°	15.62	1.562

Source: authors.

Table 3 shows a comparison of the experimental results obtained for the period T of the simple pendulum with the theoretical expressions obtained in the Section 3. The first column of Table 3 is the initial angle that the pendulum was released. The second column is the period measured with the apparatus shown in Figure 6 (the same values as in the third column of Table 2). The third column is the theoretical values obtained using Eq. (44) and the fourth column is the theoretical values obtained using Eq. (48). $L = 0{,}5\text{ m}$ and $g = 9.78\text{ m/s}^2$ were used in Eqs. (27), (45) and (48). With these values, using Eq. (27), $T_0 = 1.42068\text{ s}$. It can be seen from Table 3 that in general: $T_{\text{Eq. }48} > T_{\text{Eq. }45} > T_{\text{Eq. }27} > T_{\text{exper.}}$.

Table 3 – Experimental and theoretical results for the simple pendulum.

Oscillation amplitude (angle α)	***T*** (s) (experimental)	***T*** (s) (expansion, Eq. 45)	***T*** (s) (integral, Eq. 48)
1°	1.427	1.421	1.421
5°	1.427	1.421	1.421
10°	1.427	1.423	1.423
15°	1.430	1.427	1.427
20°	1.430	1.431	1.432
25°	1.434	1.437	1.438
30°	1.457	1.444	1.445
35°	1.461	1.453	1.454
40°	1.465	1.462	1.465
45°	1.477	1.473	1.477
50°	1.483	1.484	1.491
55°	1.505	1.496	1.507
60°	1.512	1.509	1.525
65°	1.524	1.523	1.544
70°	1.526	1.537	1.566
75°	1.530	1.552	1.590
80°	1.551	1.567	1.616
85°	1.558	1.583	1.645
90°	1.562	1.598	1.677

Source: authors.

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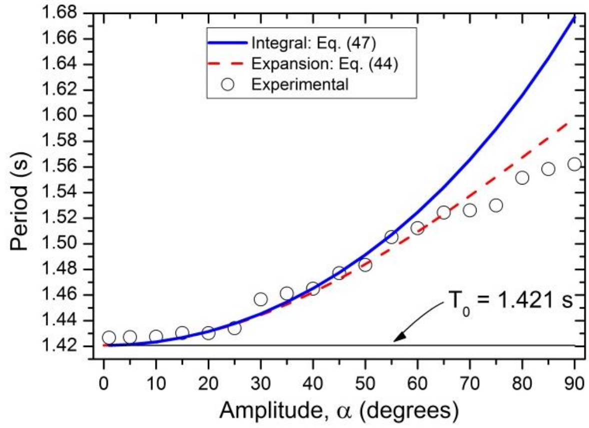
Figure 7 shows the period as a function of amplitude using the values of Table 3. The solid line was obtained using Eq. (48) and the dashed line was obtained using Eq. (45). Circles represent experimental results. Note that for angles smaller than 55º there is good agreement between experimental and theoretical results.

Figure 7 – Comparison between experimental and theoretical results of the dependence on the amplitude of the period T. In Eqs. (27), (45) and (48): $L = 0{,}5$ m and $g = 9.78\ \mathrm{m/s^2}$.



Source: authors.

FINAL COMMENTS

In this paper, the differential equations of motion that characterize and determine the motion of a simple pendulum were obtained considering small and large amplitudes of oscillation. These differential equations were solved through expansion of functions and integrations.

It was possible to verify, both experimentally and theoretically, that for the oscillatory movement of the simple pendulum, its oscillation period increases and its angular frequency decreases with the increase of the oscillation amplitude. The validity range of the approximation for small ranges of motion was also determined. It was verified that the theoretical and experimental results present a good agreement for angles smaller than 55°.

The experimental measurements were made with “a low-cost home-built” equipment. It should be noted that some factors can generate discrepancies between experimental and theoretical results. Among these factors, we can highlight: a) error in

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measuring the length of the wire; b) error in the measurement of the angle of oscillation; c) error in the measurement of time for oscillations; d) error in starting the chronometer at the exact moment when the pendulum is released and in locking the chronometer when the pendulum reaches its maximum amplitude in ten oscillations; e) air displacements occurred during the measurements; f) air resistance; g) friction at the point of support of the pendulum.

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